Solutions to "A Brief Introduction to Stochastic Calculus"

These are some solutions I have written to exercises from these popular notes from Columbia University's *IEOR E4706: Foundations of Financial Engineering*. I personally found the notes very helpful for picking up introductory stochastic calculus (e.g. Brownian motions, stochastic integrals, Itô's lemma) with minimal measure theory background.

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Exercise 1: Conditional expectations as martingales

Let Z be a random variable and set $X_t := \mathbb{E}[Z \mid \mathcal{F}_t]$. Show that X_t is a martingale.

Leaving aside the first property from Definition 3, we show the second property. Namely $\forall\ t, s \geq 0$, we have:

$$\mathbb{E}[X_{t+s} \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Z \mid \mathcal{F}_{t+s}] \mid \mathcal{F}_t]$$

but $\mathcal{F}_t \subset \mathcal{F}_{t+s}$ and so by the tower property $\mathbb{E}[X_{t+s} \mid \mathcal{F}_t] = \mathbb{E}[Z \mid \mathcal{F}_t] = X_t \implies X_t$ is a martingale.

Exercise 2: Martingale Property of Stochastic Integrals of an Elementary Process

Check that $Y_t(\omega) := \int_0^t X_s(\omega) dW_s(\omega)$ is indeed a martingale when $X_t(\omega)$ is an elementary process.

Before showing that $Y_t(\omega)$ is a martingale, we first provide our canonical definition of elementary process $X_t(\omega) := \sum_i e_i(\omega) I_{[t_i, t_{i+1})}(t)$ where $\{e_k(\omega)\}_{k=0}^n$ and $\{t_k\}_{k=0}^n$ are defined as in Definition 6.

We now check that $Y_t(\omega)$ is a martingale, this time with both properties in Definition 3. We first start by showing the more interesting second property. Pick $t, s \geq 0$. We apply Definition 7:

$$Y_{t+s}(\omega) = \int_0^{t+s} X_s(\omega) dW_s(\omega) = \sum_{i=0}^{n-1} e_i(\omega)[W_{t_{i+1} \wedge (t+s)}(\omega) - W_{t_i \wedge (t+s)}(\omega)]$$

where[a] $x \wedge y = \min(x, y)$. We now split this summation based on index j where $t_j \leq t \leq t_{j+1}$:

$$Y_{t+s}(\omega) = \underbrace{\sum_{i=0}^{j-1} e_i(\omega)[W_{t_{i+1}}(\omega) - W_{t_i}(\omega)]}_{Y_t(\omega)} + \sum_{i=j}^{n-1} e_i(\omega)[W_{t_{i+1} \wedge (t+s)}(\omega) - W_{t_i \wedge (t+s)}(\omega)]$$

$$\implies \mathbb{E}[Y_{t+s}(\omega) \mid \mathcal{F}_t] = Y_t(\omega) + \mathbb{E}[\sum_{i=j}^{n-1} e_i(\omega)[W_{t_{i+1} \wedge (t+s)}(\omega) - W_{t_i \wedge (t+s)}(\omega)] \mid \mathcal{F}_t]$$

where these expectations are over $\omega \in \Omega$. To finish, $\mathbb{E}[W_{t_{i+1}\wedge(t+s)}(\omega) - W_{t_i\wedge(t+s)}(\omega)] = 0 \implies \mathbb{E}[\sum_{i=j}^{n-1} e_i(\omega)[W_{t_{i+1}\wedge(t+s)}(\omega) - W_{t_i\wedge(t+s)}(\omega)] \mid \mathcal{F}_t] = 0$, and so we have demonstrated the martingale property $\mathbb{E}[Y_{t+s}(\omega) \mid \mathcal{F}_t] = Y_t(\omega)$.

We now show the first property for sake of completeness. We use the fact that $|\sum_i A_i| \leq \sum_i |A_i|$ where all $A_i \in \mathbb{R}$:

$$\mathbb{E}[|Y_t(\omega)|] = \mathbb{E}[|\sum_{i=0}^{n} e_i(\omega)(W_{t_{i+1}\wedge t} - W_{t_i\wedge t})|] \leq \sum_{i=0}^{n-1} \mathbb{E}[|e_i(\omega)|] \times \mathbb{E}[|W_{t_{i+1}\wedge t} - W_{t_i\wedge t}|]$$

By assumption all $|e_i(\omega)| < \infty$. Furthermore, increments in Brownian motion are normally distributed and so $W_{t_{i+1}\wedge t} - W_{t_i\wedge t} \sim \mathcal{N}(0, t_{i+1}\wedge t - t_i \wedge t)$. For any r.v. $X \sim \mathcal{N}(\mu, \sigma^2)$ we have $\mathbb{E}[|X|] = \sigma\sqrt{\frac{2}{\pi}} \implies \mathbb{E}[W_{t_{i+1}\wedge t} - W_{t_i\wedge t}] = \sqrt{\frac{2}{\pi}(t_{i+1}\wedge t - t_i \wedge t)}$. So:

$$\mathbb{E}[|Y_t(\omega)|] \leq \sqrt{\frac{2}{\pi}} \cdot \sum_{i=0}^{n-1} \mathbb{E}[|e_i(\omega)|]\sqrt{(t_{i+1}\wedge t - t_i \wedge t)} < \infty$$

which concludes our demonstration of the first property.

[a] This slight adjustment is required as $t+s$ is not necessarily equal to $T = t_n$. In the case $t+s > T$, we consider W_{t+s} as W_T – essentially stopping the Brownian motion.

Exercise 3: Prove Mixture of Independent Brownian Motions is a Brownian Motion

Let $W_t^{(1)}$ and $W_t^{(2)}$ be two independent Brownian motions. Use Levy's Theorem to show that:

$$W_t := \rho W_t^{(1)} + \sqrt{1-\rho^2}\, W_t^{(2)}$$

is also a Brownian motion for a given constant ρ.

To use Levy's Theorem (Theorem 2) to show that W_t is a Brownian motion we must show that $\forall\ T > 0, W_t$'s quadratic variation over $[0, T]$ is equal to T. We fix constants $\rho \in \mathbb{R}, T \in \mathbb{R}^+$, and make a partition $0 < t_0 < t_1 < \cdots < t_n = T$ of our interval $[0, T]$. Then we can define the sum of square changes of W_t to be $Q_n(T) := \sum_{i=1}^{n}(\Delta W_i)^2$ where each $(\Delta W_i)^2$ is given by:

$$(\Delta W_i)^2 = [W_{t_i} - W_{t_{i-1}}]^2 = [\rho(W_{t_i}^{(1)} - W_{t_{i-1}}^{(1)}) + \sqrt{1-\rho^2}(W_{t_i}^{(2)} - W_{t_{i-1}}^{(2)})]^2 = \rho^2(\Delta W_i^{(1)})^2 + (1-\rho^2)(\Delta W_i^{(2)})^2 + 2\rho\sqrt{1-\rho^2}\Delta W_i^{(1)}\Delta W_i^{(2)}$$

By Levy's Theorem, because $W_t^{(1)}$ and $W_t^{(2)}$ are Brownian Motions, their quadratic variation over interval $[0, T]$ is equal to T. So defining $\Delta t := \max_i(t_i - t_{i-1})$ we have that the quadratic variation of W_t, $\lim_{\Delta t \to 0} Q_n(T) = \lim_{\Delta t \to 0} \sum_{i=1}^{n}(\Delta W_i)^2$ is given by[a]:

$$\lim_{\Delta t\to 0} Q_n(T) = \rho^2 T + (1-\rho)^2 T + 2\rho\sqrt{1-\rho^2}\lim_{\Delta t\to 0}\sum_{i=1}^{n}\Delta W_i^{(1)}\Delta W_i^{(2)}$$
$$= T + 2\rho\sqrt{1-\rho^2}\lim_{\Delta t\to 0}\sum_{i=1}^{n}\Delta W_i^{(1)}\Delta W_i^{(2)}$$

So to show that $\lim\limits_{\Delta t\to 0} Q_n(T) = T$, we WTS that $\lim\limits_{\Delta t\to 0}\sum_{i=1}^{n}\Delta W_i^{(1)}\Delta W_i^{(2)} = 0$.
We first begin by defining r.v. $S_n := \sum_{i=1}^{n}\Delta W_i^{(1)}\Delta W_i^{(2)}$. Note that because $W_t^{(1)}$ and $W_t^{(2)}$ are independent Brownian motions, $\mathbb{E}[S_n] = \sum_{i=1}^{n}\mathbb{E}[\Delta W_i^{(1)}]\mathbb{E}[\Delta W_i^{(2)}] = \sum_{i=1}^{n} 0\cdot 0 = 0$ and so $\text{Var}(S_n) = \mathbb{E}[S_n^2]$. We look at this $\mathbb{E}[S_n^2]$ below, which will be helpful in establishing $S_n \xrightarrow{\mathbb{P}} 0$:

$$\mathbb{E}[S_n^2] = \mathbb{E}[(\sum_{i=1}^{n}\Delta W_i^{(1)}\Delta W_i^{(2)})^2] = \sum_{i,j\in[1,n]}^{n}\mathbb{E}[\Delta W_i^{(1)}\Delta W_i^{(2)}\Delta W_j^{(1)}\Delta W_j^{(2)}]$$
$$= \sum_{i,j\in[1,n]}^{n}\mathbb{E}[\Delta W_i^{(1)}\Delta W_j^{(1)}]\mathbb{E}[\Delta W_i^{(2)}\Delta W_j^{(2)}]$$

Because increments are independent in a Brownian motion $\forall i\neq j$ and $\forall k\in\{1,2\}$ we have $\mathbb{E}[\Delta W_i^{(k)}\Delta W_j^{(k)}] = \mathbb{E}[\Delta W_i^{(k)}]\mathbb{E}[\Delta W_j^{k}] = 0\cdot 0 = 0$. So we can continue simplifying $\mathbb{E}[S_n^2]$ more:

$$\mathbb{E}[S_n^2] = \sum_{i=1}^{n}\mathbb{E}[(\Delta W_i^{(1)})^2]\mathbb{E}[(\Delta W_i^{(2)})^2] = \sum_{i=1}^{n}(\Delta t_i)^2 \leq \max_i \Delta t_i \times \sum_{i=1}^{n}\Delta t_i = \max_i \Delta t_i \times T$$

But then as $\Delta t = \max\limits_i \Delta t_i \to 0$ we have $\mathbb{E}[S_n^2]\to 0$. By Chebyshev's Inequality, $\forall\epsilon > 0$:

$$\mathbb{P}(|S_n| > \epsilon) \leq \frac{\text{Var}(S_n)}{\epsilon} = \frac{\mathbb{E}[S_n^2]}{\epsilon} \to 0 \text{ as } n\to\infty$$

and so by definition of convergence in probability $S_n = \sum_{i=1}^{n}\Delta W_i^{(1)}\Delta W_i^{(2)} \xrightarrow{\mathbb{P}} 0$. Thus, $\lim\limits_{\Delta t\to 0} Q_n(T) = T \implies W_t$ is a Brownian motion by Levy's Theorem.

[a] Please note that $\Delta t\to 0 \iff n\to\infty$.