EECS 126: Probability & Random Processes **Spring 2025**

Reversible Markov Chains & Poisson Processes

1 Reversible Markov Chains

As a recap from last time on Reversible Markov Chains, remember that we have the following properties:

1. irreducible, positive recurrent Markov Chain (MC) started at stationary distribution π
2. If $(x_0, x_1, \dots x_n)$ and $(x_n, \dots, x_0)$ have the same joint PMF $\implies$ chain is "reversible"

We now present two important facts, each with their proofs:

1. **An MC summed backward is always an MC**

   Pf: Consider the following representation of a future MC, where we compute the probability of a state in the present given the future events:

   $$P(\underbrace{X_k = i}_{\text{present}} \,|\, \underbrace{X_{k+1} = j, X_{k+2} = i_{k+2}, \dots, X_n = i_n}_{\text{future}}) = \frac{P(X_k = i, X_{k+1} = j, X_{k+2} = i_{k+2}, \dots, X_n = i_n)}{P(X_{k+1} = j, X_{k+2} = i_{k+2}, \dots, X_n = i_n)}$$

   by basic conditional probability. Given our MC is starting at stationary distribution π we can re-express this probability as:

   $$\frac{\pi(i) P_{ij} P_{j,i_{k+2}} P_{i_{k+2},i_{k+3}} \dots P_{i_{n-1}}, i_n}{\pi(j) P_{j,i_{k+2}} P_{i_{k+2},i_{k+3}} \dots P_{i_{n-1},i_n}} = \frac{\pi(i) P_{ij}}{\pi(j)}$$

   Since $P(X_k | X_{k+1}, X_{k+2}, \dots, X_n)$ depends only on $P(X_k | X_{k+1})$ we have an MC! We can give the transition probability for the backward chain as:

   $$\tilde{P_{ji}} = \frac{\pi(i) P_{ij}}{\pi(j)}$$

   From this we arrive easily at our next proof.

2. **If a Backward MC is the same as its Forward MC, the MC is time-reversable**

   Pf: If a Backward MC is the same as its Forward MC $\implies \tilde{P_{ji}} = P_{ji} = \frac{\pi(i)P_{ij}}{\pi(j)} \implies \pi(i)P_{ij} = \pi(j)P_{ji}$. This means that the MC is time-reversable; the process appears the same regardless of which direction of time we are looking at. Note that $\pi(i)P_{ij} = \pi(j)P_{ji}$ is known as the *Detailed Balance Equation (DBE).* As a cool demonstration, we will briefly derive the *Global Balanced Equation (GBE)* from the DBE:

$$\pi(i)P_{ij} = \pi(j)P_{ji}$$
$$\Sigma_{i\in X}\pi(i)P_{ij} = \Sigma_{i\in X}\pi(j)P_{ji}$$
$$\Sigma_{i\in X}\pi(i)P_{ij} = \pi(j)\Sigma_{i\in X}P_{ji}$$

Note $\Sigma_{i\in X}P_{ji} = 1$ and so $\Sigma_i\pi(i)P_{ij} = \pi(j)$, which is the GBE. Expressed in matrix form, this means $\pi P = \pi$, which re-affirms that π is the stationary distribution of our MC.

We conclude this section with one last fact about MC.
Fact: Consider a graph associated with an irreducible, positive recurrent Markov chain. Construct a new graph by:

- Removing the direction of all edges in the original directed graph.
- Removing multiple edges between any pair of nodes.
- Removing all self-loops.

If the resulting undirected graph forms a tree (i.e., it contains no cycles), then the stationary distribution satisfies the DBE. *Pf*: Homework.

2 Introduction to Poisson Processes

We now shift our focus from Markov Chains to Poisson Processes. We start by defining what a Poisson Process (PP) is:

Definition 2.1. *A* ***Poisson Process (PP)*** *is the continuous-time analog of a "coin-flipping" process in the Bernoulli theorem. It serves as a fundamental model for arrival processes, such as: photons hitting a detector, packets arriving in a network, the number of accidents per day, the number of emails received per day, etc.*
Let us define T_i as the time at which the ith event occurs, and S_i as the time between T_{i-1} to T_i (we set $S_1 = T_1$). In our PP model, $S_1, S_2, \ldots, S_n \overset{I.I.D.}{\sim}$ Expo(λ) so our PDF for S_i is given by $f_{S_i}(t) = \lambda e^{-\lambda t}$ for $t > 0$.

Let us think more critically about a few things. First, note that T_n, the arrival time of the nth event, can be given as the sum of these times between events (a.k.a *interarrival times*): $T_n = \Sigma_{i=1}^n S_i$. Furthermore, if we are curious in the number of events occured by time t, denoted by N_t, we can use the following cumbersome definition:

$$N_t = \begin{cases} \max\limits_{n\geq 1}(\{n | T_n \leq t\}) & \text{if } t \geq T_1 \\ 0 & \text{else} \end{cases}$$

Before concluding our introduction to PPs for today, we try to arrive at two foundational conclusions. Before doing so, we recall some properties of Exponential distributions. Let r.v. $\tau \sim$ Expo(λ). Then:

1. Our CDF of τ is given by:

$$F_\tau(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

2. $\mathbb{E}[\tau] = \frac{1}{\lambda}$ and $\text{Var}(\tau) = \frac{1}{\lambda^2}$
3. Memoryless property: $P(\tau > t + s | \tau > s) = P(\tau > t)$
4. $P(\tau \leq t + \epsilon | \tau > t) = \lambda\epsilon + o(\epsilon)$ where $o(\epsilon)$ is "little-o" notation[1] (Direct consequence of #3)

We are specifically interested in the Memoryless property, which has the following major implication (which we will unfortunately not prove).

Theorem 2.2. *Let $t_{n+1} \geq t_n \geq 0$ be any two times in a Poisson Process. The difference $N_{t_{n+1}} - N_{t_n}$ is independent of all other increments and has the same probabilty distribution as any other difference $N_j - N_i$ where $j - i = t_{n+1} - t_n$ and $j \geq i \geq 0$. In other words, any segment of a Poisson Process is I.I.D. distributed to any other equal-length segment of the same Poisson Process.*

Finally, you may wonder why Poisson Process has the "Poisson" in its name. We address this with our final theorem which we will show next time:

Theorem 2.3. *Let $N = \{N_t, t \geq 0\}$ be a Poisson process with rate λ. Then the number of arrivals N_t in the interval $(0, t)$ follows a Poisson distribution:*

$$P(N_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!},$$

which implies that $N_t \sim Pois(\lambda t)$.

[1] $o(\epsilon)$ refers to any function that grows at a rate less than ϵ (namely $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$). One such function would be ϵ^2; this notation is used widely to address higher-order terms that vanish faster than ϵ