PTE Rick Durrett Randomly Indexed CLT

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Durrett 3.4.6. *Let $X_1, X_2, \dots$ be i.i.d with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = \sigma^2 \in (0, \infty)$, and let $S_n = X_1 + \cdots + X_n$. Let N_n be a sequence of nonnegative integer-valued random variables and a_n a sequence of integers with $a_n \to \infty$ and $N_n/a_n \to 1$ in probability. Show that*

$$S_{N_n}/\sigma\sqrt{a_n} \Rightarrow \chi$$

We first aim to show $\frac{S_{N_n}}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}} \xrightarrow{p} 0$. We are first going to fix $\epsilon > 0$. First observe that for any $\delta > 0$.

$$\{|S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n}\} \subseteq \{|S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n}, |N_n - a_n| \leq \delta a_n\} \cup \{|N_n - a_n| > \delta a_n\}$$

So we can write:

$$\mathbb{P}(|S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n}) \leq \mathbb{P}(S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n}, |N_n - a_n| \leq \delta a_n) + \mathbb{P}(|N_n - a_n| > \delta a_n)$$

We first try to understand the first term. Recall that $\forall c \geq d, S_c - S_d \stackrel{d}{=} S_{c-d}$ and Kolmogorov's maximal inequality:

$$\mathbb{P}(|S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n}, |N_n - a_n| \leq \delta a_n) \leq \mathbb{P}(\max_{1 \leq k \leq \delta a_n} |S_{\min(N_n, a_n)+k} - S_{\min(N_n, a_n)}| \geq \epsilon\sigma\sqrt{a_n})$$
$$= \mathbb{P}(\max_{1 \leq k \leq \delta a_n} |S_k| \geq \epsilon\sigma\sqrt{a_n}) \leq \frac{\sigma^2 \cdot \delta a_n}{\epsilon^2\sigma^2 a_n} = \frac{\delta}{\epsilon^2}$$

So the upper bound for $\mathbb{P}(|S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n})$ for any chosen $\delta > 0$ can be given as:

$$\mathbb{P}(|S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n}) \leq \frac{\delta}{\epsilon^2} + \mathbb{P}(|N_n - a_n| > \delta a_n)$$

We will first send $n \to \infty$. Observe that $N_n/a_n \xrightarrow{p} 1 \implies \mathbb{P}(|N_n - a_n| > \delta a_n) \to 0$. Thus for a fixed $\delta > 0$, sending $n \to \infty$ yields:

$$\limsup_{n \to \infty} \mathbb{P}(|S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n}) \leq \frac{\delta}{\epsilon^2}$$

Moreover, now sending $\delta \downarrow 0$ does not change the LHS but makes the RHS bound go to zero. But $\liminf_{n\to\infty} \mathbb{P}(|S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n}) \geq 0$ holds trivially and so:

$$\mathbb{P}(|S_{N_n} - S_{a_n}| \geq \epsilon\sigma\sqrt{a_n}) \to 0 \text{ as } n \to \infty$$

But because ϵ was arbitrary $\implies \frac{S_{N_n}}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}} \xrightarrow{p} 0$. By normal CLT, we know $\frac{S_{a_n}}{\sigma\sqrt{a_n}} \Rightarrow \chi$ so applying Slutsky's with the previous convergence result yields $\frac{S_{N_n}}{\sigma\sqrt{a_n}} \Rightarrow \chi$ as desired.