Solutions to Cornell/Bard "Lebesque Measure" Notes

These are some solutions I have written to exercises from these notes from Cornell University / Bard College's course on measure theory taught by Dr. Jim Belk. I found the notes and exercises to be very helpful.

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Exercise 1

If $\{E_n\}$ is a sequence of measurable sets, prove that the intersection $\bigcap_{n\in\mathbb{N}} E_n$ is measurable.

$\bigcap_{n\in\mathbb{N}} E_n = (\bigcup_{n\in\mathbb{N}} E_n^c)^c$. Since all $\{E_n^c\}$ measurable, $\bigcup_{n\in\mathbb{N}} E_n^c$ measurable $\implies$ $(\bigcup_{n\in\mathbb{N}} E_n^c)^c$ is measurable.

Exercise 2

Prove that if $S \subseteq \mathbb{R}$ and $m^*(S) = 0$, then S is measurable.

Fix some subset $S' \subseteq \mathbb{R}$. Note that for any collection of open intervals $\mathcal{C}$ of S', $\sum_{I\in\mathcal{C}} \ell(I) \geq 0 \implies m^*(S') \geq 0$.

Pick any test subset $E \subseteq \mathbb{R}$. Then $E \cap S \subset S \implies m^*(E \cap S) \leq m^*(S) \leq 0$. But $m^*(E \cap S) \geq 0 \implies m^*(E \cap S) = 0$. By identical logic, $m^*(E^c \cap S) = 0$ and so we have:

$$m^*(E \cap S) + m^*(E^c \cap S) = 0 = m^*(S)$$

Exercise 3

a) *If $E \subseteq F$ are measurable sets, prove that $F - E$ is measurable.*

b) *Prove that if $m(E) < \infty$ then $m(F - E) = m(F) - m(E)$.*

Part (a). $F - E = F \cap E^c$, which is measurable.
Part (b). Define $\{A_n\}$ to be a sequence of pairwise disjoint measurable subsets of $\mathbb{R}$ where $A_1 = F - E, A_2 = E$ and $\forall k \geq 3, A_k = \emptyset$. Note that $F = \biguplus_{k\in\mathbb{N}} A_n$ and so:

$$m(F) = m(\biguplus_{k\in\mathbb{N}} A_n) = m(F - E) + m(E) + 0 \implies m(F - E) = m(F) - m(E)$$

Exercise 4

If E and F are measurable sets with finite measure, prove that

$$m(E \cup F) = m(E) + m(F) - m(E \cap F)$$

$E \cap (F \cap E^c) = \emptyset \implies m(E \cup F) = m(E \cup (F \cap E^c)) = m(E) + m(F \cap E^c) = m(E) + m(F) - m(E \cap F)$.

Exercise 5

Suppose that $E \subseteq S \subseteq F$, where E and F are measurable. Prove that if $m(E) = m(F)$ and this measure is finite, then S is measurable as well.

Note that because $E \subseteq S \subseteq F \implies m^*(E) \leq m^*(S) \leq m^*(F)$. But since $m^*(E) = m^*(F) \implies m^*(S) = m^*(E) = m^*(F)$. Now pick test subset $T \subseteq \mathbb{R}$. Note the following two equations:

$$m^*(E \cap T) \leq m^*(S \cap T) \leq m^*(F \cap T)$$

and similarly:

$$m^*(E \cap T^c) \leq m^*(S \cap T^c) \leq m^*(F \cap T^c)$$

Adding them together, we have $m^*(E) \leq m^*(S \cap T) + m^*(S \cap T^c) \leq m^*(F) \implies m^*(S \cap T) + m^*(S \cap T^c) = m^*(S) \implies S$ is measurable.

Exercise 6

Prove that every countable subset of $\mathbb{R}$ is measurable and has measure zero.

As a hint, note that this statement is true for every finite subset of $\mathbb{R}$. Pick any countable subset $S \subseteq \mathbb{R}$. In view of Exercise 2, it is sufficient to show $m^*(S) = 0$. Pick any $\epsilon > 0$. We now define a collection of open intervals $\mathcal{C}$ to cover S where for the kth point in S (denoted by S_k), the kth (open) interval in $\mathcal{C}$ is given by $(S_k - \frac{\epsilon}{2^{k+1}}, S_k + \frac{\epsilon}{2^{k+1}})$. Then we have:

$$\sum_{I \in \mathcal{C}} \ell(I) = \sum_{k=1} \frac{\epsilon}{2^k} = \epsilon$$

But because ϵ was arbitrary we have $m^*(S) \leq 0$. But $m^*(S) \geq 0 \implies m^*(S) = 0$, and so we are finished.

Exercise 7

Given a nested sequence $E_1 \subseteq E_2 \subseteq \ldots$ of measurable sets, prove that

$$m(\bigcup_{n \in \mathbb{N}} E_n) = \sup_{n \in \mathbb{N}} m(E_n)$$

Note that this proof is really the general case proof of measures being continuous from above/below. Also note that $\uparrow$ means convergence from above.

Because $\{E_n\}$ is a non-decreasing sequence of sets, $\{m(E_n)\}$ is also a non-decreasing sequence $\implies m(\bigcup_{n\in\mathbb{N}} E_n) = \sup_{n\in\mathbb{N}} m(E_n) \iff m(E_n) \uparrow m(\bigcup_{n\in\mathbb{N}} E_n) \iff \forall\epsilon > 0, \exists\, k$ s.t. $m(E_k) > m(\bigcup_{n\in\mathbb{N}} E_n) - \epsilon$. We prove this last statement.

We first start by defining the following:

$$A_1 := E_1, \quad A_{n+1} := E_{n+1} - E_n$$

where $\{A_n\}$ is clearly a sequence of pairwise disjoint measurable sets and $E_k = \bigcup_{n=1}^k A_n$ and $\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} A_n$. For convenience, we can define $E := \bigcup_{n\in\mathbb{N}} E_n$. We proceed by defining the "partial sum of measurable sets" $S_k := \sum_{n=1}^k m(A_n) = m(E_k)$. Note $\{S_k\}$ and $\{m(E_n)\}$ are non-decreasing sequences (so their limits are equal to their supremum) and thus:

$$m(E) = \sum_{n=1}^{\infty} m(A_n) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} m(E_n) = \sup_{n\in\mathbb{N}} m(E_n)$$

Exercise 8

a) *Let $E_1 \supseteq E_2 \supseteq \dots$ be a nested sequence of measurable sets with*

$$\bigcap_{n\in\mathbb{N}} E_n = \emptyset$$

Prove that if $m(E_1) < \infty$, then $m(E_n) \to 0$ as $n \to \infty$.

b) *Let $E_1 \supseteq E_2 \supseteq \dots$ be a nested sequence of measurable sets, and suppose that $m(E_1) < \infty$. Prove that*

$$m(\bigcap_{n\in\mathbb{N}} E_n) = \inf_{n\in\mathbb{N}} m(E_n)$$

c) *Give an example of a nested sequence $E_1 \supseteq E_2 \supseteq \dots$ of measurable sets such that $m(E_n) = \infty$ for all n but*

$$m(\bigcap_{n\in\mathbb{N}} E_n) < \infty$$

Part (a). Note that $E_1 - E_n \uparrow E_1$ and so by Exercise 7, $m(E_1 - E_n) \uparrow m(E_1)$. Note that because $\forall n, E_1 \supseteq E_n \implies m(E_1 - E_n) = m(E_1) - m(E_n)$ (see Exercise 3b, note $E_n \subseteq E_1 \implies m(E_n) \leq m(E_1) < \infty$). Thus we have:

$$m(E_1 - E_n) \uparrow m(E_1) \implies m(E_1) - m(E_n) \uparrow m(E_1) \implies m(E_n) \downarrow 0$$

Part (b). Note that $\bigcap_{n\in\mathbb{N}} E_n = E_1 - \bigcup_{n\in\mathbb{N}}[E_1 - E_n]$. Using Exercise 7 and Exercise 3(b), we have:

$$\begin{aligned} m(\bigcap_{n\in\mathbb{N}} E_n) &= m(E_1 - \bigcup_{n\in\mathbb{N}}[E_1 - E_n]) = m(E_1) - m(\bigcup_{n\in\mathbb{N}}[E_1 - E_n]) \\ &= m(E_1) - \sup_{n\in\mathbb{N}}[m(E_1) - m(E_n)] = -\sup_{n\in\mathbb{N}}[-m(E_n)] = \inf_{n\in\mathbb{N}} m(E_n) \end{aligned}$$

Part (c). Define each $E_n = (n, \infty)$. While each $m(E_n) = \infty$, $\bigcap_{n\in\mathbb{N}} E_n = \emptyset \implies m(\bigcap_{n\in\mathbb{N}} E_n) = m(\emptyset) = 0 < \infty$.